Теория игр и устройство рынков

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game.hse.ru

- Теория игр
- Устройство рынков

- Теория игр
  - зачем?
  - что?
  - как?

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- Устройство рынков
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Теория игр (некооперативная)

- N – множество игроков
- S_i – множество стратегий игрока $\mathrm{i} \in N$
  - $S = S_1 \times S_2 \times \cdots \times S_i$ -- множество всех векторов стратегий
  - $s \in S$ – профиль стратегий
- u_i – функция полезности (выигрыши), переводит профиль стратегий в число
  - чем число больше, тем лучше

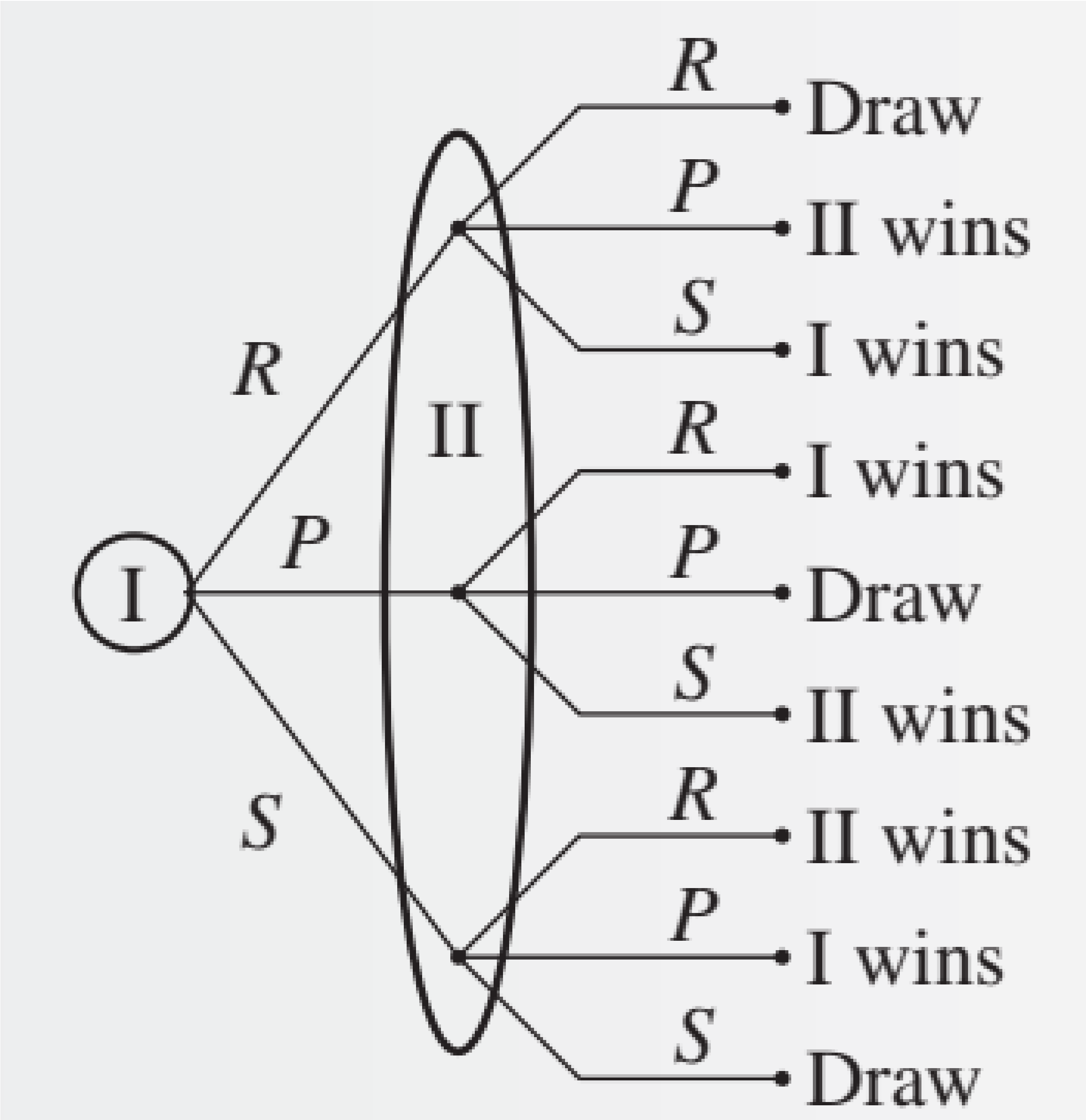
$$u_i : S \rightarrow \mathbb{R}$$

  - u - профиль полезностей (предпочтений)
- Игра: (N, S, u)
- Что дальше?

		Player II		
		Rock	Paper	Scissors
Player I	Rock	0, 0	1, –1	1, –1
	Paper	1, –1	0, 0	– 1, 1
	Scissors	–1, 1	–1, 1	0, 0

прочие усложнения

- Как добавить случайность?
- ……………………одновременность?


R
Draw
P
II wins
S
I wins
R
II
R
I wins
P
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прочие усложнения

- Как добавить случайность?
- ……………………одновременность?
- …………………… неопределённость относительно действий?
- …………………… неопределённость относительно выигрышей?
- …………………… ожидания относительно выигрышей (типов)?
- ?

концепции решения

что будет происходить?

		Player II		
		L	*M*	*R*
Player I	*T*	1, 0	1, 2	0, 1
	B	0, 3	0, 1	2, 0

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	L	*M*
T	1, 0	1, 2

Игра «угадай 2/3 среднего»

- Конкурс красоты Кейнса (на самом деле, Эрве Мулена)

- Выберите число от 0 до 1000
- Победит тот, чьё число ближе всех к 2/3 среднего от всех чисел

концепции решения: доминирование

- стратегия s_i является доминирующей если для любых s_i',s_{-i} верно:

$$u_i(s_i,s_{-i})>=u_i(s_i',s_{-i})$$

Пример: аукцион второй цены

- An object is for sale
- There are N buyers, each buyer i has a value v_i
- (each value is a private information, values are iid on R_+)

The auction rules are:

- each buyer j submits a private bid b_j
- the buyer with the highest bid receives the object $b_i = \max_{j \in N} b_j .$
- (if several bids win, then pick among these bidders at random)
- and pays not his bid but the second-highest bid $\max_{j \neq i} b_j$

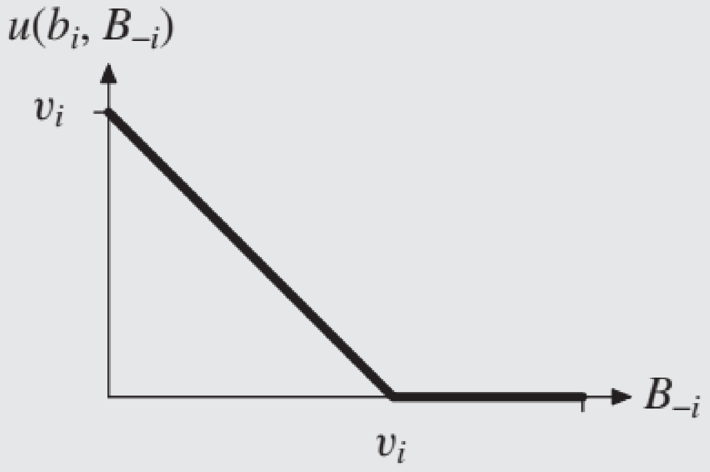
The game:

1. The set of players is the set N of buyers in the auction.
2. The set of strategies available to buyer i is the set of possible bids $S_i = [0, \infty)$.
3. The payoff to buyer i, when the strategy vector is $b = (b_1, \ldots, b_n)$, is

$$u_i(b) = \begin{cases} 0 & \text{if } b_i < \max_{j \in N} b_j, \\ \frac{v_i - \max_{j \neq i} b_j}{|\{k : b_k = \max_{j \in N} b_j\}|} & \text{if } b_i = \max_{j \in N} b_j. \end{cases}$$

Пример: аукцион второй цены

the highest bid except i's bid: $B_{-i} = \max_{j \neq i} b_j$
Compare the utility of these strategies depending on B_{-i}



The payoff function for strategy $b_i = v_i$

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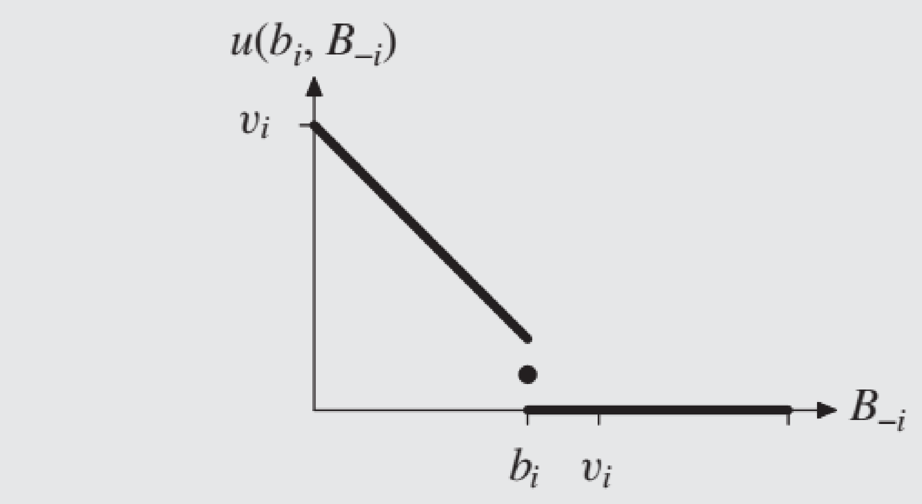

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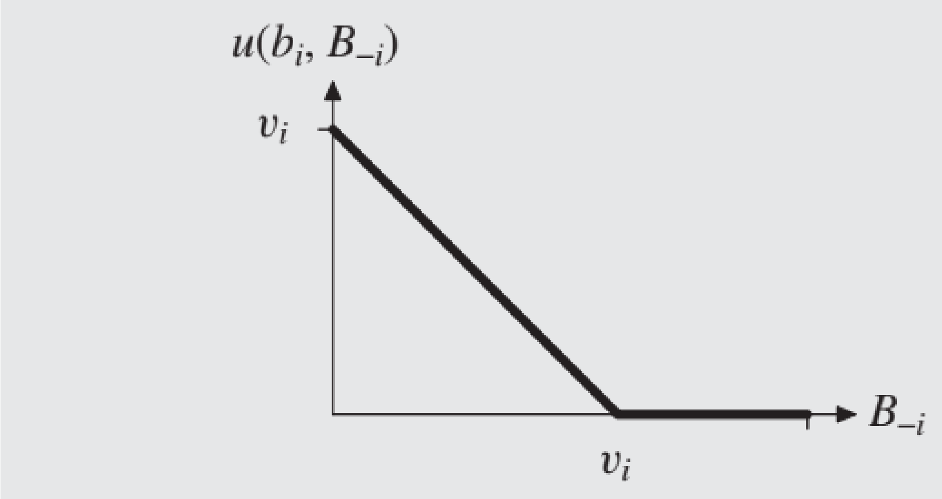

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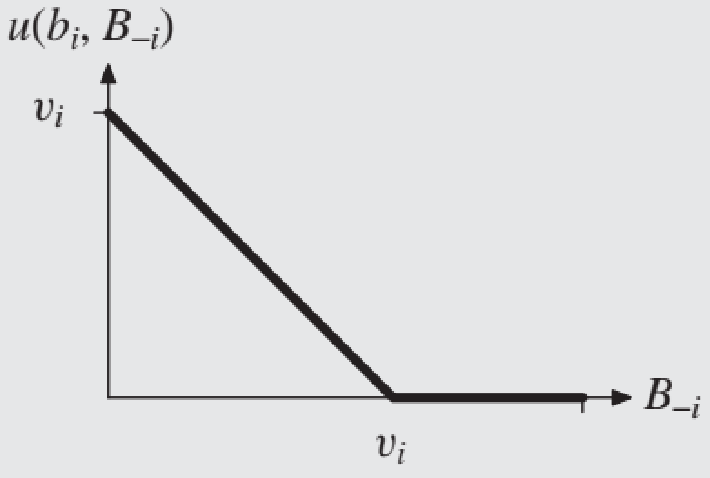

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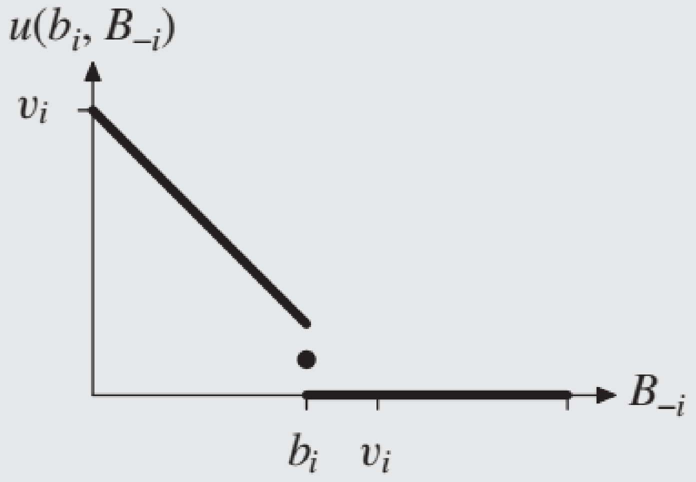

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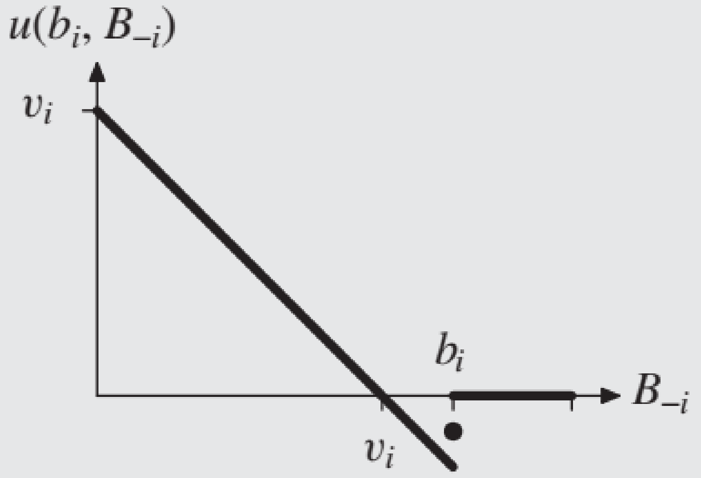

The payoff function for strategy $b_i > v_i$

Равновесие Нэша

- равновесная стратегия $s^* =$ оптимальный ответ на себя

$$s_i^* \in argmax(u_i(s_i^*, s_{-i}^*)$$

- «никто не хочет отклониться от равновесной стратегии»

- расширим стратегии на симплекс:
  - каждому игроку разрешим играть (чистые) стратегии с вероятностью
  - из множества равновесий S_i получаем M_i
  - полезность от $m \in M$ будет мат.ожиданием

$$u_i(m) \equiv \sum_{s \in S} m_1(s_1) \cdots m_N(s_N) u_i(s)$$

Оно существует!

- В любой конечной игре существует равновесие Нэша (в см.стр.)

  - M – выпуклый компакт
  - Для любой стратегии m, игрока i и его чистой стратегии j:

$$f_{ij}(m) = \frac{m_{ij} + \max(0, u_i(j, m_{-i}) - u_i(m))}{1 + \sum_{j'=1}^{n} \max(0, u_i(j', m_{-i}) - u_i(m))}$$

  - (f – тоже смешанная стратегия)
  - => существует неподвижная точка, она и будет равновесием!

резюме

- теория игр – основа общественных наук
  - как действуют агенты при заданных правилах?
- определение игры
- концепции решения
- существования равновесия Нэша

- открытые вопросы остаются как в самой теории,
- так и в приложениях: в решении обратной задачи:
  - как создать «хорошие» правила игры?
  - голосование, трудовые контракты, законы (конституция!), делёж…
- и основная задача это…

Устройство рынков

Что такое рынки?

- это то, где спрос встречается с предложением
  - ярмарка, овощебаза, авито, hh.ru, летняя стажировка лаборатории
- не все рынки одинаковы
  - большинство работает на основе ценового механизма
    - продавец назначает цену, покупатель сравнивает и выбирает
  - и этого достаточно (фундаментальные результаты об эффективности)
  - но есть особые рынки:
    - они не будут эффективными (нуждаются в регулировании)
    - или даже будут разваливаться!!!

Introduction

- Markets seem to be easy, self-governing and efficient
- Oftentimes markets need a careful and informed design
- Otherwise, markets can *suffer* from adverse competition,
    - lead to *bad outcomes* and even *unravel*
- Case-study:
  Evolution of the labor market for medical interns and residents
    - Early 20 century: introduction of internships
    - Students were interested in experience, hospitals – in cheap labor
    - From the beginning, there were more vacancies than candidates
    - Hospitals needed to compete for interns
    - One cheap form of competition was the time of hiring:
        - initially and optimally: end of 4th year

Evolution of the labor market for medical interns

- 1927: Letter to the Association of American Medical Colleges
  - "For a number of years **attempts have been made to defer the appointment of hospital interns until towards the close of the fourth year**. The Association of American Medical Colleges, the Council on Medical Education of the American Medical Association, and the American Hospital Association **have all passed resolutions favoring this idea**. **The difficulty has been in persuading someone to take the lead.**
  - This is to inform you that it has been decided to defer the appointments of interns at the Presbyterian Hospital in the City of New York until some time **in April**.
  - It is earnestly hoped that other hospitals and schools will be able to act in a similar manner."
- 1939: most appointments were made during **December,** but many earlier(including the Presbyterian Hospital)
- 1944: — 2 full years before the internship, or even earlier!
- The outcome of such competition is detrimental for all parties
- What exactly is the problem? **Free-riding on public good**
- **Solution**: schools won't release transcripts until senior year

Evolution of the labor market for medical interns

- The market became **thick**: large demand, large supply
- However, a new problem appeared in the final stage:
- Hospitals post ads, students apply, get offers, what next?
  - Student gets acceptable offer and wait-listed at better hospital
  - if he accepts, but the later better offer comes, he is unhappy
  - if he accepts, but reneges for better offer, hospital is unhappy
  - if he rejects, and better offer never comes, both are unhappy
- 1945: "Students should be allowed 10 days to consider"
- 1946: Transcripts released on June 1, offers sent on July 1, decisions after July 8
  - 25% of hospitals violated the rules
- By 1949: proposed appointments made at 12.01 AM on November 15 by telegram
  - and applicants have 12 hours to consider;
  - but hospitals rejected 12 hours waiting period and
  - also noted that filing telegrams in advance to be delivered on 12.01 is OK
- 1950: 12 hour period was back (no calls or telegrams allowed)

Evolution of the labor market for medical interns

- What exactly is the problem? **Congestion**
- Application-rejection process does not solve the issue regardless of timing
- They lacked **tools** to arrive at **satisfactory outcome**
- 1950/51 - trial run using a new centralized procedure
- 1951/52 - National Intern Matching Program, voluntary participation
  - 95% of eligible students and hospitals chose to participate
  - (high participation continued until 1970s, redesigned in 1998)
- How did this work?
- What outcome is satisfactory? What are the tools to reach it?

One-to-one matching: the marriage market

- A **marriage market** (Gale and Shapley, 1962) is a triple (M, W, P) s.t.
  1. $M = \{m_1, ..., m_p\}$ is a set of men
  2. $W = \{w_1, ..., w_q\}$ is a set of women
  3. $P = (P_k)_{k \in M \cup W}$ is a preference profile, (R is a weak extension) where
     - 3.1 each man $m \in M$ has a strict preference P_m over $W \cup \{m\}$ (being matched with a woman or being single)
     - 3.2 each man $w \in W$ has a strict preference P_w over $M \cup \{w\}$ (being matched with a man or being single)

  - Let $\mathcal{P}_m$ be the set of all preferences of man m, $\mathcal{P}_w$ the set of all preferences of woman w; thus $\mathcal{P} = \times_{k \in M \cup W} \mathcal{P}_k$

Example (Example 2.9 in Roth and Sotomayor, 1990)

- $M = \{m_1, m_2, ..., m_5\}$, $W = \{w_1, w_2, ..., w_4\}$
- (in Roth and Sotomayor (1990) R denotes strict preferences, but we stick with P)

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{m_5}
w_1	w_4	w_4	w_1	w_1
w_2	w_2	w_3	w_4	w_2
w_3	w_3	w_1	w_3	w_4
w_4	w_1	w_2	w_2	m_5
m_1	m_2	m_3	m_4	

R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}
m_2	m_3	m_5	m_1
m_3	m_1	m_4	m_4
m_1	m_2	m_1	m_5
m_4	m_4	m_2	m_2
m_5	m_5	m_3	m_3
w_1	w_2	w_3	w_4

Definitions

- Woman w is **acceptable** to man m (at P_m) if $wP_m m$
- Man m is **acceptable** to woman w (at P_w) if $mP_w w$
- A **matching** μ is a bijection: $M \cup W \to M \cup W$ such that for each $m \in M$ and each $w \in W$,
  - $\mu(m) \in W \cup \{m\}$
  - $\mu(w) \in M \cup \{w\}$
  - $\mu(m) = w \iff \mu(w) = m$ – man m is matched to woman w (and t.o.w.a.)
  - $\mu(m) = m$ – man m is unmatched (single)
  - let $\mathcal{M}$ be the set of all matchings

Individual rationality and stability

- A matching μ is **blocked by an individual** $k \in M \cup W$ at P
  - if k prefers being single to being matched with $\mu(k)$: $kP_k\mu(k)$

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  - if it is not blocked by any individual agent
  - i.e. each matched partner is acceptable
- A matching μ is **blocked by a pair** $(m, w) \in M \times W$ at P
  - if they prefer each other to their partners at μ: $wP_m\mu(m)$ and $mP_w\mu(w)$

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- A matching μ is **stable** at P
  - if it is not blocked by any individual and any pair.
- A matching μ is **in the (weak) core**
  - if there exist no matching ν and coalition $T \subseteq M \cup W$
  - such that for each $k \in T, \nu(k) \in T$ and $\nu(k)P_k\mu(k)$

Motivating stability and the core

- For the marriage market stability means fidelity
- For practical applications stability is crucial
  - nobody is able to profit from changing the matching individually or as a pair
  - (the chance of such a change makes agents seek blocking pairs)
  - in case of centralized market, agents might prefer to contract outside of the market
  - competitive market equilibrium (with prices) is also stable
- The core: nobody is able to profit from changing the matching in any coalition

Stability and the core: equivalence

- **Theorem**: for each $P \in \mathcal{P}$, the set of stable matchings equals the core

Stability and the core: equivalence

- **Theorem**: for each $P \in \mathcal{P}$, the set of stable matchings equals the core
- Proof:
- 1. Each matching in the core is stable:
  - obvious, as single agent or a pair are coalitions
- 2. Each stable matching is in the core:
  - proof by contradiction: let μ be stable but not in the core
  - then, by definition, there exist matching ν and coalition $T \subseteq M \cup W$
  - such that for each $k \in T, \nu(k) \in T$ and $\nu(k) P_k \mu(k)$
  - k is better off at ν: $\nu(k) P_k \mu(k)$
  - and k's new partner $\nu(k)$ is better off at ν: $k P_{\nu(k)} \mu(\nu(k))$
  - thus k and $\nu(k)$ block μ and μ is not stable! Contradiction.
  - Therefore: (each stable matching) μ is in the core.

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- Does a stable matching always exist? Is there a systematic way (mechanism) to find one?

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- Does a stable matching always exist? Is there a systematic way (mechanism) to find one?
- The answer is affirmative: DA

Men-Proposing Deferred Acceptance Algorithm (DA)

- We fix $P \in \mathcal{P}$
  - **Step1.** Each man m proposes to his 1st choice (if he has any acceptable mates). Each woman rejects any unacceptable proposals and, if more than one acceptable proposal is received, holds the most preferred and rejects the rest.

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  - **Step2.** Each rejected man, proposes to his 2nd choice...
  - **Step k.** Any man who was rejected at step $k-1$ makes a new proposal to his most preferred acceptable woman who has not rejected him yet (If no acceptable women remain, he makes no proposal.). Each woman holds her most preferred acceptable offer, and rejects the rest.

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- By switching the roles of men and women, we can define the **women-proposing DA algorithm**.

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- The algorithm terminates after a step where no rejections are made by matching each woman to the man (if any) whose proposal she is "holding".
- By switching the roles of men and women, we can define the **women-proposing DA algorithm**.
- We denote by $DA^M(P)$ the matching selected by men-proposing DA algorithm, and by $DA^W(W)$ the matching selected by women-proposing DA algorithm.

Example

-

R_{m_1}	R_{m_2}	R_{m_3}	R_{m_4}	R_{m_5}
w_1	w_4	w_4	w_1	w_1
w_2	w_2	w_3	w_4	w_2
w_3	w_3	w_1	w_3	w_4
w_4	w_1	w_2	w_2	m_5
m_1	m_2	m_3	m_4	

R_{w_1}	R_{w_2}	R_{w_3}	R_{w_4}
m_2	m_3	m_5	m_1
m_3	m_1	m_4	m_4
m_1	m_2	m_1	m_5
m_4	m_4	m_2	m_2
m_5	m_5	m_3	m_3
w_1	w_2	w_3	w_4

Example: solution

1. $m_1 - w_1, m_2 - w_4$
2. $m_5 - w_2, m_3 - w_3, m_4 - w_4$
3. $m_2 - w_2$
4. $m_4 - w_4$

- m_5 is rejected by all acceptable women and remains single. Thus the algorithm ends.
- the matching is: $DA^M(P) = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 & m_5 \\ m_1 & m_2 & m_3 & m_4 & m_5 \end{pmatrix}$

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- is it stable? how about WPDA matching?
- Men prefer MPDA over WPDA, women – opposite
- m_5 is unmatched under both MPDA and WPDA

Further examples

- Let all men have the same preferences over women. What is the MPDA outcome? Is it stable?

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- Proof: fix $P \in \mathcal{P}$, we show that $DA^M(P)$ is stable at P
  - observe that in men-proposing DA, in each step women get weakly better off and men get weakly worse off.
  - (it has to terminate since preferences are finite)
  - **individual rationality.** Matching $DA^M(P)$ cannot be blocked by any individual since men do not propose to unacceptable women and women reject unacceptable men.

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    - Thus $DA^M_w(P)P_w m$ – woman w prefers her DA-partner to m,
    - which contradicts that (m, w) blocks $DA^M(P)$.
    - Thus, $DA^M(P)$ is stable at P.

Set of stable matchings

- **Corollary**: DA^W is also stable.
- How different are these two stable matchings?
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- This result can be generalized

Set of stable matchings: optimal matchings

- Def: a stable matching μ is **men-optimal stable matching (MOSM)** if for each other stable matching μ', all men (weakly) prefer μ over μ'
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  - then (m', w) block μ, and thus μ is not stable!?!
  - thus, at DA^M each man gets best ach.wmn, => =MOSM.

Matching mechanisms

- A **(direct) mechanism** is a function from profiles to matchings $\varphi : \mathcal{P} \mapsto \mathcal{M}$
- Note:
  - we assume that there is a central authority which collects preferences and announces the matching that is induced by the mechanism at these preferences
  - thus, each mechanism induces a preference revelation game, where each agent's strategy set is the set of her preferences
- A mechanism φ is **individually rational** if for each preference profile P it induces an individually rational at P matching $\varphi(P)$
- A mechanism φ is **stable** if for each preference profile P it induces a stable at P matching $\varphi(P)$
- A mechanism φ is **strategy-proof** if for each preference profile P truth-telling is a weakly dominant strategy to each agent $i \in M \cup W$
  - for each P_i' , $\varphi_i(P)R_i\varphi_i(P_i', P_{-i})$.

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    - w_1 reports m_1 as unacceptable and gets m_2 who is better for her
  - if the mechanism induces WOSM, who can manipulate and how?
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Strategic behavior for men/women

- The mechanism is **strategy-proof for men** if no man can manipulate it.

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- **Theorem (Dubins and Freedman 1981; Roth 1982)**: MOSM is strategy-proof for men.

Рынок знакомств

Целевой приём

проблемы на рынке труда для молодых

- разрушение рынков
- корпоративные университеты (центральный университет)
- ранний найм, совмещение
- олимпиады для школьников
- равномерное распределение
- двухэтапная карьера

- решение: целевой приём (сейчас 150/600тыс. мест – целевой резерв)

Model

- I – set of **students**, typical student $i \in I$
- s – **school** (focus on one school)
- F – set of **firms**; typical firm $f \in F$; dummy $f_0 \in F$ – ordinary seats

- each student $i \in I$ has preferences P_i over $F \cup \emptyset$, profile P_I

- s has total capacity q_s
- $r \leq q_s$ seats are reserved for targeted admission
- s has strict preference P_s over $2^{I \cup \emptyset}$

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- ...(dummy firm f_0 coincides with s, $P_{f_0} = P_s$, $q_{f_0} = q_s$)

- $P = (P_I, P_s, P_F)$ – preference profile of all agents
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- (I, s, F, q, r, P) – targeted reserve problem

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Резюме

- устройство рынков (и вообще правил игры)
  - полезно
  - интересно

- приходите к нам учиться и работать
- game.hse.ru