Web graph models: properties and applications

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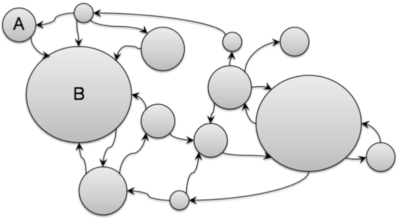
World-wide web graph



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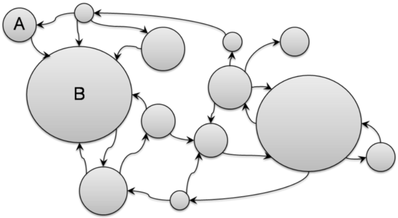

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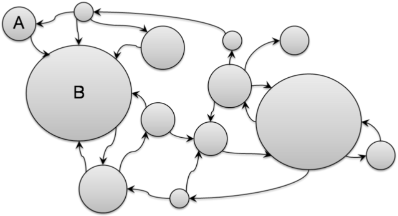
World-wide web graph



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- Sparse graphs (n vertices, mn edges)

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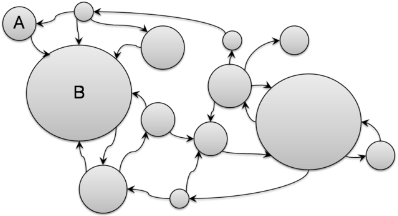

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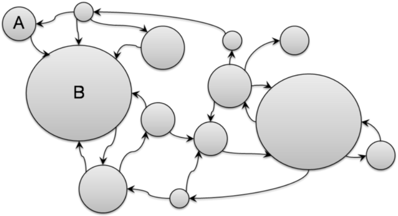
World-wide web graph

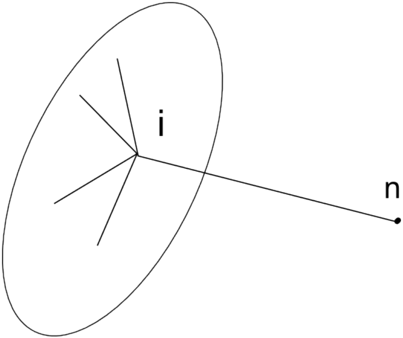


Experimental observations [Barabási, Albert (1999)]:

- Sparse graphs (n vertices, mn edges)
- Small world (diameter $\approx$ 5-7)
- Power law

$$\frac{|\{v : \mathrm{deg}(v) = d\}|}{n} \approx \frac{c}{d^\lambda}, \quad \lambda \sim 2.1$$

At the n-th step we add a new vertex n with m edges from it, with probability of edge to a vertex i proportional to deg(i)



$$\mathbf{P}(\text{edge from } n \text{ to } i) = \frac{\deg(i)}{\sum_j \deg(j)}$$

Problems with formalization when $m > 1$

Theorem (Bollobás)

Let $f(n)$, $n \geq 2$, be any integer valued function with $f(2) = 0$ and $f(n) \leq f(n+1) \leq f(n) + 1$ for every $n \geq 2$, such that $f(n) \to \infty$ as $n \to \infty$. Then there is a random graph process of Barabási and Albert $T^{(n)}$ such that, with probability 1, $T^{(n)}$ has exactly $f(n)$ triangles for all sufficiently large n.

$G_m^{(n)}$ – graph with n vertices and mn edges, $m \in \mathbb{N}$.
$d_G(v)$ – degree of vertex v in graph G.

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Case m =1

$G_1^{(1)}$ – graph with one vertex v_1 and one loop.
Given $G_1^{(n-1)}$ we can make $G_1^{(n)}$ by adding vertex v_n and edge from it to vertex v_i, picked from $\{v_1, \ldots, v_n\}$ with probability

$$\mathbf{P}(i = s) = \begin{cases} d_{G_1^{(n-1)}}(v_s)/(2n-1) & 1 \le s \le n-1 \\ 1/(2n-1) & s = n \end{cases}$$

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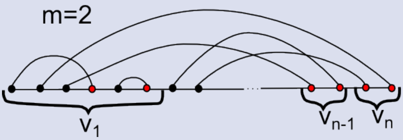
Case m >1

Given $G_1^{(mn)}$ we can make $G_m^{(n)}$ by gluing $\{v_1, \dots, v_m\}$ into v_1' , $\{v_{m+1}, \dots, v_{2m}\}$ into v_2', and so on.

Linearized chord diagrams (LCD) model

LCD with $2mn$ vertices and mn edges.



Denote by $\phi(L)$ graph obtained after gluing.
If L is chosen uniformly from all $\frac{(2mn)!}{(mn)!2^{mn}}$ LCDs with mn edges, then $\phi(L)$ has the same distribution as $G_m^{(n)}$.

Theorem (Bollobás–Riordan–Spencer–Tusnády)

Let $m \geq 1$ and $\epsilon \geq 0$ be fixed, and set

$$\alpha_{m,d} = \frac{2m(m+1)}{(d+m)(d+m+1)(d+m+2)}.$$

Then **whp** we have

$$(1-\epsilon)\alpha_{m,d} \leq \frac{\#_m^n(d)}{n} \leq (1+\epsilon)\alpha_{m,d}$$

for every d in the range $0 \leq d \leq n^{\frac{1}{15}}$

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Fix an integer $m \geq 2$ and a positive real number ϵ. Then **whp** $G_m^{(n)}$ is connected and has diameter diam $\left(G_m^{(n)}\right)$ satisfying

$$(1-\epsilon)\log n/\log\log n \leq \text{diam}\left(G_m^{(n)}\right) \leq (1+\epsilon)\log n/\log\log n$$

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Theorem (Grechnikov)

$$E(\#_m^n(d)) = I\{d \geq 0\}\frac{(2mn+1)(m+1)}{(d+m)(d+m+1)(d+m+2)} - \frac{I\{d=0\}}{m} + O_m\left(\frac{d}{n}\right)$$

$I\{X\}$ — indicator of event X.

Theorem (Grechnikov)

If $d = d(n)$ and $\psi(n) \to \infty$ when $n \to \infty$, then **whp** we have

$$\left|E(\#_m^n(d)) - \#_m^n(d)\right| \leq \left(\sqrt{d^{-3}n} + d^{-1}\right)\psi(n)$$

Case m = 1

For a fixed positive integer a define a process $H_{a,1}^{(n)}$ exactly as $G_1^{(n)}$ is defined above, but replacing probability of edge with

$$\mathbf{P}(i = s) = \begin{cases} \frac{d_{H_{a,1}^{(n-1)}}(v_s)+a-1}{(a+1)n-1} & 1 \le s \le n-1 \\ \frac{a}{(a+1)n-1} & s = n \end{cases}$$

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Case m > 1

As for $G_m^{(n)}$, a process $H_{a,m}^{(n)}$ is defined by identifying vertices in groups of m.

Theorem (Buckley–Osthus)

Let $m \geq 1$ and $a \geq 1$ be fixed integers, and set

$$\alpha_{a,m,d} = (a+1)(am+a)! \binom{d+am-1}{am-1} \frac{d!}{(d+am+a+1)!}.$$

Let $\epsilon > 0$ be fixed. Then **whp** we have

$$(1-\epsilon)\alpha_{a,m,d} \leq \frac{\#^n_{a,m}(d)}{n} \leq (1+\epsilon)\alpha_{a,m,d}$$

for all d in the range $0 \leq d \leq n^{\frac{1}{100}(a+1)}$. In particular, **whp** for all d in this range we have

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Let $a > 0$ be fixed real, then

$$E\left(\#_{a,m}^{n}(d)\right) = \frac{\mathrm{B}(d+ma, a+2)}{\mathrm{B}(ma, a+1)} n + O_{a,m}\left(\frac{1}{d}\right)$$

The asymptotic behavior of the coefficient when d grows is

$$\frac{\mathrm{B}(d+ma, a+2)}{\mathrm{B}(ma, a+1)} \sim \frac{\Gamma(a+2)}{\mathrm{B}(ma, a+1)} d^{-2-a} = (a+1)\frac{\Gamma(ma+a+1)}{\Gamma(ma)} d^{-2-a}$$

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Let $d_1 > 0$ and $d_2 > 0$. Then

$$\mathrm{cov}(\#^n_{a,m}(d_1), \#^n_{a,m}(d_2)) = O_{a,k}((d_1^{-2-a} + d_2^{-2-a})n + d_1^{-1}d_2^{-1})$$

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Theorem (Grechnikov)

If $d = d(n)$ and $\psi(n) \to \infty$ when $n \to \infty$, then **whp** we have

$$\left| \#_{a,m}^{n}(d) - \frac{\mathrm{B}(d+ma, a+2)}{\mathrm{B}(ma, a+1)} n \right| \leq \left(\sqrt{d^{-a-2} n} + d^{-1}\right) \psi(n)$$

Consequences

When $d \sim Cn^{\frac{1}{a+2}}$ with some constant C,

$$E\left(\#^n_{a,m}(d)\right) = O(1), \quad \sqrt{d^{-a-2}n} + d^{-1} = O(1).$$

If

$$d = o\left(n^{\frac{1}{a+2}}\right),$$

then **whp**

$$\#^n_{a,m}(d) \sim \frac{(a+1)\Gamma(ma+a+1)}{\Gamma(ma)} d^{-2-a} n.$$

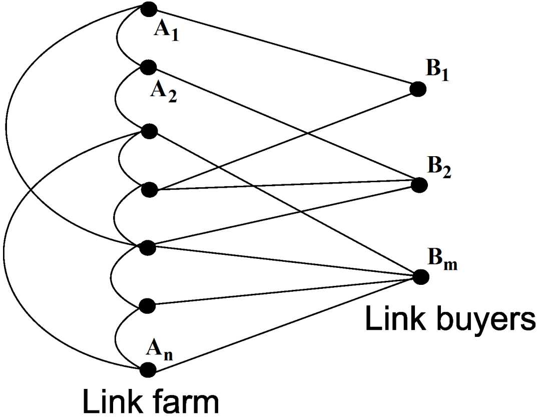
If

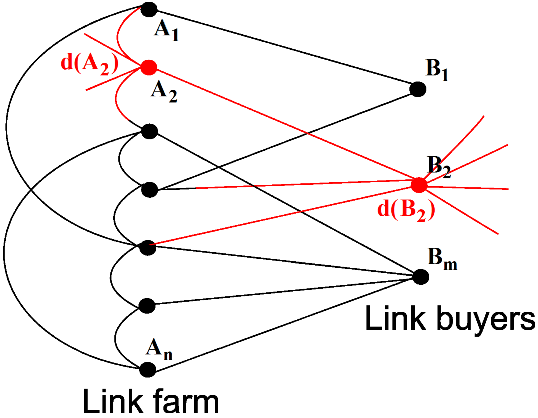
$$d = \omega\left(n^{\frac{1}{a+2}}\right),$$

then **whp** $\#^n_{a,m}(d) = o(1)$; since $\#^n_{a,m}(d)$ is an integer number by definition, in this case **whp**

$$\#^n_{a,m}(d) = 0.$$

Link rings





N = number of edges between farm and buyers

$X(d_1, d_2, n)$ – total number of edges linking a node with degree d_1 and a node with degree d_2. When $d_1 = d_2$, we count every edge twice, but do not count loops.

The expected value for N given $d(A_i)$ and $d(B_j)$ is

$$N_0 = \sum_{i=1}^{n} \sum_{j=1}^{m} X(d(A_i), d(B_j), n).$$

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- If $N \leq N_0$, this structure can be a natural formation.
- If $N > N_0$, this structure is probably a real link farm with some buyers.

Theorem (Grechnikov)

There exists a function $c_X(d_1, d_2)$ such that

$$EX(d_1, d_2, n) = c_X(d_1, d_2)n + O_{a,m}(1).$$

When both d_1 and d_2 grow, the asymptotic behaviour of c_X is

$$c_X(d_1, d_2) = ma(a+1)\frac{\Gamma(ma+a+1)}{\Gamma(ma)}\frac{(d_1+d_2)^{1-a}}{(d_1)^2(d_2)^2}\cdot$$
$$\cdot\left(1 + O_{a,m}\left(\frac{1}{d_1} + \frac{1}{d_2} + \frac{d_1 d_2}{(d_1+d_2)^2}\right)\right).$$

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Theorem (Grechnikov)

Let $c > 0$. Then

$$P\left(|X(d_1, d_2, n) - EX(d_1, d_2, n)| \geq c(d_1+d_2)\sqrt{mn}\right) \leq 2\exp\left(-\frac{c^2}{8}\right).$$

In particular, if $c(n) \to \infty$ when $n \to \infty$, then **whp** $|X - EX| < c(n)(d_1+d_2)\sqrt{mn}$

Total number of edges between vertices with fixed degree

The formula for $c_X(d_1, d_2)$ does not give an asymptotic behaviour if

$$\frac{d_2}{d_1} \to c \neq 0.$$

The precise bounds show that the term

$$\frac{(d_1 + d_2)^{1-a}}{d_1^2 d_2^2}$$

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and

$$c_X(d_1, d_2) = \frac{\Gamma(d_1 + ma)\Gamma(d_2 + ma)\Gamma(d_1 + d_2 + 2ma + 3)}{\Gamma(d_1 + ma + 2)\Gamma(d_2 + ma + 2)\Gamma(d_1 + d_2 + 2ma + a + 2)} \cdot$$
$$\cdot\, ma(a+1)\frac{\Gamma(ma + a + 1)}{\Gamma(ma)}\left(1 + \theta(d_1, d_2)\frac{(d_1 + ma + 1)(d_2 + ma + 1)}{(d_1 + d_2 + 2ma + 1)(d_1 + d_2 + 2ma + 2)}\right)$$

where

$$-4 + \frac{2}{1 + ma} \leq \theta(d_1, d_2) \leq a\frac{\Gamma(ma + 1)\Gamma(2ma + a + 3)}{\Gamma(2ma + 2)\Gamma(ma + a + 2)}$$

Theorem (Grechnikov)

If $d_1 + d_2 = 0$, then $X(d_1, d_2, n) = 0$. If $d_1 + d_2 \geq 1$, then

$$EX(d_1, d_2, n) = \frac{m(m+1)}{(d_1+m)(d_1+m+1)(d_2+m)(d_2+m+1)} \cdot \left(1 - \frac{C_{2m+2}^{m+1} C_{d_1+d_2}^{d_1}}{C_{d_1+d_2+2m+2}^{d_1+m+1}}\right)(2mn+1) - \sum_{i=1}^{m} \frac{C_{d_1+d_2+2m-2i}^{d_1+m-i}}{(d_1+m)(d_2+m)C_{d_1+d_2+2m}^{d_1+m}} \left(\frac{(2i)!}{i!(i+1)!}\frac{m+1}{2m} + [i=m]\frac{(2m)!}{2(m-1)!^2}\right) - [d_1=0]\frac{(m-1)(m+1)}{2m(d_2+m)(d_2+m+1)} - [d_2=0]\frac{(m-1)(m+1)}{2m(d_1+m)(d_1+m+1)} + O_{m,d_1,d_2}\left(\frac{1}{n}\right).$$

Define 2-nd degree of vertex t as

$$d_2(t) = \#\{i, j : i \neq t, j \neq t, it \in E(G_1^{(n)}), ij \in E(G_1^{(n)})\}$$

Define by $X_n(k)$ number of vertices with 2-nd degree equal to k in $G_1^{(n)}$

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$$d_2(t) = \#\{i,j : i \neq t, j \neq t, it \in E(G_1^{(n)}), ij \in E(G_1^{(n)})\}$$

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Theorem (Grechnikov–Ostroumova)

For any $k \geq 1$

$$E\left(X_n(k)\right) = \frac{4n}{k^2}\left(1 + O\left(\frac{k^2}{n}\right)\right)\left(1 + O\left(\frac{\log^2 k}{k}\right)\right).$$

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Theorem (Ostroumova)

For any $\varepsilon > 0$ there is such a function $\varphi(n) = o(n)$, that for any $1 \leq k \leq n^{1/6-\varepsilon}$, **whp** we have

$$\left|X_n(k) - E\left(X_n(k)\right)\right| \leq \frac{\varphi(n)}{k^2}$$

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Theorem about pairs of adjacent edges (P_2) (Bollobás)

$$(1-\epsilon)\frac{m(m+1)}{2} n \ln n \leq \#(P_2, G_m^{(n)}) \leq (1+\epsilon)\frac{m(m+1)}{2} n \ln n$$

holds **whp** as $n \to \infty$ where $\epsilon > 0$ be fixed

Theorem about arbitrary subgraph (Ryabchenko – Samosvat)

for arbitrary graph H

$$\mathbf{E}\left(\#\left(H, G_m^{(n)}\right)\right) = \Theta(1)\left(n^{\#(d_i=0)}(\sqrt{n})^{\#(d_i=1)}(\ln n)^{\#(d_i=2)}\right) m^{\left(\frac{\sum d_i}{2}\right)}$$

where d_i — is degree of node i in H

or

$$\mathbf{E}\left(\#\left(H, G_m^{(n)}\right)\right) = 0$$

$\alpha, \beta, \gamma, \delta_{\text{in}}, \delta_{\text{out}}$ — parameters,
$x_i(n)$ — a number of vertices with indegree i,
$y_i(n)$ — a number of vertices with outdegree i.
Let

$$c_1 = \frac{\alpha + \beta}{1 + \delta_{\text{in}}(\alpha + \gamma)}, c_2 = \frac{\beta + \gamma}{1 + \delta_{\text{out}}(\alpha + \gamma)}.$$

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Theorem

Let $i \geq 0$. There exists p_i, q_i such that **whp** $x_i(n) = p_i n + o(n)$, $y_i(n) = q_i n + o(n)$.
If $\alpha\delta_{\text{in}} + \gamma > 0$, $\gamma < 1$ then

$$p_i \sim C_{\text{in}} i^{-X_{\text{in}}}$$

as $i \to \infty$, where $X_{\text{in}} = 1 + \frac{1}{c_1}$, C_{in} — some positive constant.
If $\gamma\delta_{\text{out}} + \alpha > 0$, $\alpha < 1$ then

$$q_i \sim C_{\text{out}} i^{-X_{\text{out}}}$$

as $i \to \infty$, where $X_{\text{out}} = 1 + \frac{1}{c_2}$, C_{out} — some positive constant.

Future Research

- Consideration of all discussed properties in the directed model. We want to obtain the same asymptotic and concentration results.

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- Studying of the distribution of second degrees in the Bollobás–Riordan model with $m \geq 2$.
- Consideration of the second degree distribution in all discussed models.