EECS 495: Combinatorial Optimization **Lecture 6**
Matroids

Reading: Schrijver, Chapter 39

Matroids

[[*Abstracts linear algebra and graph theory.*]]
Key set systems to keep in mind:

- subsets of vectors of $\mathcal{R}^n$
- subsets of edges of $G = (V, E)$

Def: A *matroid* $M = (\mathcal{S}, \mathcal{I})$ is a finite ground set $\mathcal{S}$ together with a collection of sets $\mathcal{I} \subseteq 2^{\mathcal{S}}$ satisfying:

- downward closed: if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$, and
- exchange property: if $I, J \in \mathcal{I}$ and $|J| > |I|$, then there exists an element $z \in J \setminus I$ s.t. $I \cup \{z\} \in \mathcal{I}$.

Terminology:

- $I \in \mathcal{I}$ *independent*, $I \notin \mathcal{I}$ *dependent*
- *circuit* is a minimal dependent set of M
- *basis* is a maximal independent set
- I is a *spanning set* if for some basis B, $B \subseteq I$

Example: *Uniform matroids* U_n^k: Given by $|S| = n$, $\mathcal{I} = \{I \subseteq S : |I| \leq k\}$.

Check two properties and see this is a matroid.

What are the...

- bases: sets of size k
- circuits: sets of size $k+1$
- spanning sets: sets of size at least k

Example: *Linear matroids*: Let F be a field, $A \in F^{m \times n}$ an $m \times n$ matrix over F, $S = \{1, \ldots, n\}$ be index set of columns of A. Then $I \subseteq S$ is independent if the corresponding columns are linearly independent.

Check two properties and see this is a matroid.

What are the...

- bases: minimal sets of vectors that span space spanned by A
- circuits: vectors that span space space spanned by A with one extra
- spanning sets: vectors that span space spanned by A

Note: Linear matroids can be representated as:

$$A = [I_m | B]$$

since

- If not full row rank, can remove redundant rows, and

- get above form with row operations and column swaps.

Example: *Graphic Matroids*: Let $G = (V, E)$ be a graph and $S = E$. A set $F \subseteq E$ is independent if it is acyclic.

Check two properties and see this is a matroid.

What are the...

- bases: minimum spanning trees
- circuits: subgraphs with one cycle
- spanning sets: connected subgraphs that contain every vertex

Example: *Matching Matroids*: The matching matroid $M = (V, \mathcal{I})$ for graph $G = (V, E)$ has $U \subseteq V$ independent if there's a matching in G that covers all of U.

Check two properties and see this is a matroid. For exchange,

- Consider $I, J \in \mathcal{I}$ with $|I| < |J|$.
- Let M_I, M_J be matchings for I, J and suppose M_I doesn't cover anything in $J \setminus I$.
- Consider matching defined by symmetric diff of M_I and M_J.
- Note each $v \in J \setminus I$ starts an alternating path.
- Some such paths don't end in $I \setminus J$ since $|J \setminus I| > |I \setminus J|$. Let P be one such path.
- P doesn't end in $J \cap I$ since those vertices have degree 0 or 2, so P ends not in I.
- Now M_I symmetric diff with P is a matching that covers all of I and one extra vertex in $J \setminus I$.

What are the...

- bases: minimum spanning trees
- circuits: subgraphs with one cycle
- spanning sets: connected subgraphs that contain every vertex

Note: All bases of a matroid M must have same cardinality.

Def: The *rank function* of M is $r : 2^S \to \mathcal{Z}_+$ given by $r(U) = \max_{I \subseteq U, I \in \mathcal{I}} |I|$.

Note: Corresponds to rank of matrix in linear matroids, hence name.

Def: (Alternate defn of matroid): $M = (S, \mathcal{I})$ is a matroid if there's a rank function $r : 2^S \to \mathcal{Z}_+$ such that

- $r(U) \subseteq |U|$ for all U,
- *monotonicity*: $T \subseteq U \to r(T) \leq r(U)$,
- *submodularity*: $\forall A, B \subseteq S, r(A \cap B) + r(A \cup B) \leq r(A) + r(B)$ (equivalently, $\forall C \subseteq D, \forall j \notin D, r(D \cup \{j\}) - r(D) \leq r(C \cup \{j\}) - r(C)$),

in which case we can take $\mathcal{I} = \{U : r(U) = |U|\}$.

Duality

Def: Given matroid $M = (S, \mathcal{I})$, the dual matroid $M^* = (S, \mathcal{I}^*)$ is defined by $\mathcal{I}^* = \{I \subseteq S | S \setminus I \text{ is a spanning set of } M\}$.

Note: $(M^*)^* = M$.

Claim: M^* is a matroid.

Proof: Clearly downward closed. For exchange, consider $I, J \in \mathcal{I}^*$ with $|I| < |J|$.

- $S \setminus J$ contains base B of M

- then $B \setminus I \subseteq B' \subseteq S \setminus I$ for some basis B'
- and $J \setminus I \not\subseteq B'$ since otherwise (as $B \cap I \subseteq I \setminus J$ and $(B \setminus I) \cap (J \setminus I) = \emptyset$):

$$|B| = |B \cap I| + |B \setminus I|$$

$$\leq |I \setminus J| + |B \setminus I|$$

$$< |J \setminus I| + |B \setminus I|$$

$$\leq |B'|$$

  contradicting all bases have same size.

- thus $\exists z \in J \setminus I$ with $z \notin B'$ so $I \cup \{z\} \in \mathcal{I}^*$.

Claim: The rank function r_{M^*} satisfies $r_{M^*}(U) = |U| + r_M(S \setminus U) - r_M(S)$.

Proof: Let $\mathcal{B}$ and $\mathcal{B}^*$ denote collections of bases of M and M^*. Then:

$$r_{M^*}(U) = \max_{A \in \mathcal{B}^*}\{|U \cap A|\} = \max_{B \in \mathcal{B}}\{|U \setminus B|\}$$

$$= |U| - \min_{B \in \mathcal{B}}\{|B \cap U|\}$$

$$= |U| - r_M(S) + \max_{B \in \mathcal{B}}\{|B \setminus U|\}$$

$$= |U| - r_M(S) + r_M(S \setminus U).$$

Example: Graphic matroid.

- Dual is: set of edges that when removed leave graph connected.
- Dual is graphic iff graph is planar,
- in which case dual is graphic matroid of planar dual.

Representation

Def: For a field F, a matroid M is *representable* over F if it can be expressed as a linear matroid with matrix A and linear independence taken over F.

Example: Uniform matroid U_4^2 not binary:

- if so, would have matrix with columns 1/2 being $(0,1)$ and $(1,0)$ and remaining two vectors with entries in $0,1$ neither all zero.
- only three such non-zero vectors, so can't have all pairs indep.

Question: representation of U_4^2? $(1,0),(0,1),(1,-1),(1,1)$ in $\Re$.

Def: A *binary* matroid is a matroid representable over $GF(2)$.

Def: A *regular* matroid is representable over any field.

Example: Graphic matroids are regular.

Proof: Take A to be vertex/edge incidence matrix with $+1/-1$ in each column in any order.

- Minimally dependent sets sum to zero perhaps with multiplying by -1.
- Works over any field with $+1$ as multiplicative identity and -1 additive inverse of $+1$.

Note: so far have graphic $\subset$ binary $\subset$ regular $\subset$ linear.