EECS 495: Combinatorial Optimization	**Lecture 8**
Matroid Intersection	

Reading: Schrijver, Chapter 41

Matroid Intersection

Problem:

- Given: matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ on *same* ground set S
- Find: max weight (cardinality) common independent set $J \subseteq \mathcal{I}_1 \cap \mathcal{I}_2$

Applications

Generalizes:

- max weight independent set of M (take $M = M_1 = M_2$)
- matching in bipartite graphs

  For bipartite graph $G = ((V_1, V_2), E)$, let $M_i = (E, \mathcal{I}_i)$ where

  $$\mathcal{I}_i = \{J : \forall v \in V_i, v \text{ incident to} \leq \text{one } e \in J\}$$

  Then

  - these are matroids (partition matroids)
  - common independent sets = matchings of G
- colorful spanning forests: for graph G with edges partitioned into color classes $\{E_1, \ldots, E_k\}$, colorful spanning forest is a forest with edges of different colors.

  Define M_1, M_2 on ground set E with:

  - $\mathcal{I}_1 = \{F \subset E : F \text{ is acyclic}\}$
  - $\mathcal{I}_2 = \{F \subset E : \forall i, |F \cap E_i| \leq 1\}$

  Then

  - these are matroids (graphic, partition)
  - common independent sets = colorful spanning forests

Note: In second two examples, $(S, \mathcal{I}_1 \cap \mathcal{I}_2)$ not a matroid, so more general than matroid optimization.

Claim: Matroid intersection of three matroids NP-hard.

Proof: Reduction from directed Hamiltonian path: given digraph $D = (V, E)$ and vertices s, t, is there a path from s to t that goes through each vertex exactly once.

- M_1 graphic matroid of underlying undirected graph
- M_2 partition matroid in which $F \subseteq E$ indep if each v has at most one incoming edge in F, except s which has none
- M_3 partition ..., except t which has none

Intersection is set of vertex-disjoint directed paths with one starting at s and one ending at t, so Hamiltonian path iff max cardinality intersection has size $n-1$.

Min-Max Theorem

Natural upper bound: Let $J \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $A \subseteq S$. Then

$$J \cap A \in \mathcal{I}_1 \text{ and } J \cap \overline{A} \in \mathcal{I}_2$$

so

$$|J| = |J \cap A| + |J \cap \overline{A}| \leq r_1(A) + r_2(\overline{A})$$

Claim: (Edmonds, 1970) For matroids M_1, M_2 on S,

$$\max_{J \in \mathcal{I}_1 \cap \mathcal{I}_2} \{|J|\} = \min_{A \subseteq S} \{r_1(A) + r_2(\overline{A})\}.$$

Applications:

- generalizes Konig (in bipartite graphs, max matching = min vertex cover):

  Since $r_i(F)$ is number of $v \in V_i$ covered by F,

  $$\nu(G) = \min_{F \subseteq E}(r_1(F) + r_2(E \setminus F)).$$

  as $v \in V_1$ covered by F plus $v \in V_2$ covered by $E \setminus F$ form a vertex cover.

- generalizes Konig-Rado (in bipartite graphs, max indep set equals min edge cover):

  Edge cover is

  $$\min_{F \text{ spans } M_1, M_2} |F| = \min_{B_i \text{ basis of } \mathrm{M_i}} |B_1 \cup B_2|$$

  $$= \min_{B_i \text{ basis of } \mathrm{M_i}} (|B_1| + |B_2| - |B_1 \cap B_2|)$$

  $$= r_1(E) + r_2(E) - \min_{F \subseteq E}(r_1(F) + r_2(E \setminus F))$$

  which is number of vertices minus min vertex cover = max indep set.

  [[*General statement about spanning sets in matroids.*]]

- gives nec and suff conditions for existence of colorful spanning tree:

  Want

  $$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = |V| - 1,$$

  or equivalently

  $$\min_{F \subseteq E}(r_1(F) + r_2(E \setminus F)) = |V| - 1.$$

  Note

  $$r_1(F) = |V| - c(F)$$

  where $c(F)$ is num connected comp of $G' = (V, F)$, so need

  - num colors in $E \setminus F$ at least $c(F) - 1$
  - iff removing any t colors leaves at most $t+1$ conn comp

Proof

Need:

1. deletion
2. contraction
3. submodularity of rank function

Def: A function f is submodular if for any A, B,

$$f(A \cup B) + f(A \cap B) \leq f(A) + r(B)$$

or equivalently if for any $S \subset T$ and $i \notin T$,

$$f(S \cup \{i\}) - f(S) \geq f(T \cup \{i\}) - f(T).$$

[[*Compare to concavity.*]]

Claim: Above defns equivalent.

Proof:

$\leftarrow$: For A, B, if $B \subseteq A$, claim trivial. Else let $B \setminus A = \{b_1, \ldots, b_k\}$

$$\begin{aligned} f(A \cup B) - f(A) &= \sum_{i=1}^{k} (f(A + b_1 + \ldots + b_i) \\ &\quad -f(A + b_1 + \ldots + b_{i-1})) \\ &\leq \sum_{i=1}^{k} (f(A \cap B + b_1 + \ldots + b_i) \\ &\quad -f(A \cap B + b_1 + \ldots + b_{i-1})) \\ &= f(B) - f(A \cap B) \end{aligned}$$

$\rightarrow$: For $S \subseteq T$, $i \notin T$, set $A = S \cup \{i\}$ and $B = T$:

$$\begin{aligned} f(T+i) + f(S) &= f(A \cup B) + f(A \cap B) \\ &\leq f(A) + f(B) \\ &= f(S+i) + f(T) \end{aligned}$$

Claim: $r(\cdot)$ is rank func of a matroid iff

- $r(\emptyset) = 0$ and $r(A \cup \{e\}) - r(A) \in \{0, 1\}$ for all e, A
- $r(\cdot)$ is submodular

Proof:

$\leftarrow$: First condition obvious. For second, fix $A \subseteq C$, $e \notin C$. Want:

$$r(C \cup \{e\}) - r(C) \leq r(A \cup \{e\}) - r(A).$$

- since $r(A \cup \{e\}) - r(A) \geq 0$ and $r(C \cup \{e\}) - r(C) \in \{0, 1\}$, may assume $r(C \cup \{e\}) - r(C) = 1$
- let J_A, J_C be max indep sets in A, C with $J_A \subseteq J_C$
- since $r(C \cup \{e\}) = r(C) + 1$ know $J_C \cup \{e\} \in \mathcal{I}$ (exchange property)
- since $J_A \subseteq J_C$ know $J_A \cup \{e\} \in \mathcal{I}$ (downward closure)
- therefore $r(A \cup \{e\}) = r(A) + 1$

$\rightarrow$: exercise.