$\min_{x \in F_{st}} \|x\|_\infty$

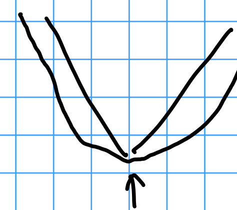
$F_{st} = \{ x \mid \underbrace{Bx}_{\text{demands}} = \chi_{s,t} \}$

$\min_{x \in F_{st}} \text{smax}_\varepsilon(x)$

$\text{smax}_\varepsilon(x) = \ln\left(\frac{\sum_i e^{\frac{x_i}{\varepsilon}} + e^{-\frac{x_i}{\varepsilon}}}{2m} \right)$

$\text{smax}_\varepsilon(\ )$ is $\frac{1}{\varepsilon}$-smooth

$\|x\|_\infty \geq \text{smax}_\varepsilon(x) \geq \|x\|_\infty - \varepsilon \ln 2m$



$|\cdot|$

Laplacian

Elect. flows

= linear sys

Max flow via GD:

① x^0

② For $i = 1, \dots, T$:

$x^{i+1} \leftarrow \Pi\left(x^i - \eta \nabla f(x^i)\right)$

$f(x) = \text{smax}_\varepsilon(x)$

$\Pi(y) = \arg\min_{z \in F_{st}} \|z - y\|$

Can show: $T \approx \frac{\beta \cdot \|x^0 - x^*\|_2^2}{\varepsilon} \Rightarrow f(x^T) - f(x^*) \leq \varepsilon$

For maxflow: $\beta = \frac{1}{\varepsilon}$ $\quad \|x^0 - x^*\|_2^2 \leq O(n)$

$\Rightarrow$ Runtime: $\tilde{O}\left(\frac{n}{\varepsilon^2} \cdot m\right) = \boxed{\tilde{O}\left(\frac{mn}{\varepsilon^2}\right)}$

$\tilde{O}(n^{\frac{3}{2}})$ $\quad$ $\Downarrow$ $\tilde{O}\left(\frac{n^2}{\varepsilon^2}\right)$

$\min_x f(x)$

$y = x + \delta$

$f(y) \leq \underbrace{f(x) + \nabla f(x)^T \delta}_{} + \overbrace{\frac{\beta}{2} \|\delta\|_2^2}^{}$

$\delta = -\frac{1}{\beta} \nabla f(x)$

$f(x) - f(x - \frac{1}{\beta} \nabla f(x))$

$\geq \frac{1}{2\beta} \|\nabla f(x)\|_2^2$

$$\min_x f(x)$$

$$y = x + \delta$$

$$\boxed{f(y) = f(x) + \nabla f(x)^T \delta + \frac{\beta}{2}\|\delta\|^2}$$

Def. f is β smooth wrt $\|\cdot\|$

$$\forall_{x,y} \quad \|\nabla f(x) - \nabla f(y)\|_* \leq \beta \|x-y\|$$

$$\boxed{\|y\|_* = \min_{x, \|x\| \leq 1} x^T y}$$

- $\|x\|_A = \sqrt{x^T A x} \Rightarrow \|\cdot\|_{A^{-1}}$
- $\|\cdot\|_2 \Rightarrow \|\cdot\|_2$
- $\|\cdot\|_\infty \Rightarrow \|\ \|_1$
- $\|\cdot\|_1 \Rightarrow \|\ \|_\infty$
- $\|\cdot\|_p \Rightarrow \|\ \|_q$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$f(y) = f(x) + \underbrace{\nabla f(x)^T \delta}_{(*)} + \underbrace{\frac{\beta}{2}\|\delta\|^2}_{(**)}$$

$$\delta = -(\nabla f(x))^\#$$

$$y^\# = \underset{u}{\arg\max}\ \underbrace{y^T u}_{(*)} - \underbrace{\frac{1}{2}\|u\|^2}_{(**)}$$

$\|\ \| \leftarrow L_2$ norm

$$y^\# = y$$

- $L_\infty \quad y^\#_i = \text{sign}(y_i)\,|y|_1$
- $\|\ \|_A \quad y^\# = A^{-1} y$

since ε is $\frac{1}{\varepsilon}$-smooth wrt $\|\cdot\|_\infty$

General GD

1. x^0
2. $x^{i+1} \leftarrow \Pi\left(x^i - \frac{1}{\beta}(\nabla f(x^i))^\#\right)$

$$\Pi(y) = \underset{x \in S}{\arg\min} \|x - y\|$$

$$\Rightarrow \text{RUNTIME: } O\left(\frac{\overbrace{\beta}^{\frac{1}{\varepsilon}} \cdot \|x^0 - x^*\|_\infty^2}{\varepsilon}\right)$$

$\Rightarrow$ Max flow $\Rightarrow$ $O(\frac{1}{\varepsilon^2})$ iteration

$$\Pi_\infty(x) = \underset{y \in F_{s,t}}{\operatorname{argmin}} \|x-y\|_\infty$$

2 electrical flows

∞ max flow

$\Pi_\infty(0)$ = max flow of the graph

Idea: Solve this problem α-approx. max flow

$O\left(\frac{\alpha \cdot \beta \cdot \|x^0 - x^*\|_\infty^2}{\varepsilon}\right) \Rightarrow$

$\frac{\alpha}{F^*}$ $\alpha \gg 1$

$\Rightarrow O(\frac{1}{\varepsilon^2} \cdot n^{o(1)})$ iterations

each iter. $O(m^{1+o(1)})$ time $\tilde{O}(m^{1+o(1)})$ $\boxed{\alpha = n^{o(1)}}$

$\Rightarrow O\left(\frac{1}{\varepsilon^2} m^{1+o(1)}\right) \rightarrow (1+\varepsilon)$-approx. max flow.

$\downarrow$

$\tilde{O}(\frac{1}{\varepsilon} m)$

exact solution $\varepsilon = \frac{1}{m}$

exact directed max flow

$\Rightarrow$ exact undir. max flow

Goal: Get better accuracy

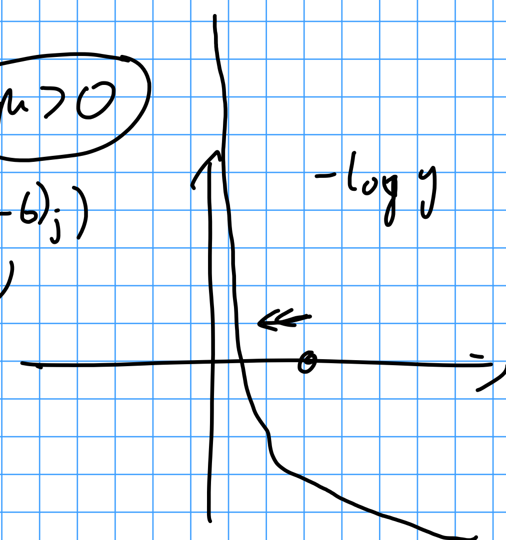
Interior-Point Method(s)

Way to solve a Linear program (LP)

$$LP \quad \min_x \; c^T x \quad \text{s.t.} \quad Ax \geq b$$

$$\mu > 0$$

$$LP(\mu) = \min_x \; c^T x - \underbrace{\mu \sum_i \log((Ax-b)_i)}_{\text{barrier term}}$$



(1) For μ suff. large

$x^*(\mu)$ = opt. sol. $LP(\mu)$

Can find $x^*(\mu)$ easily

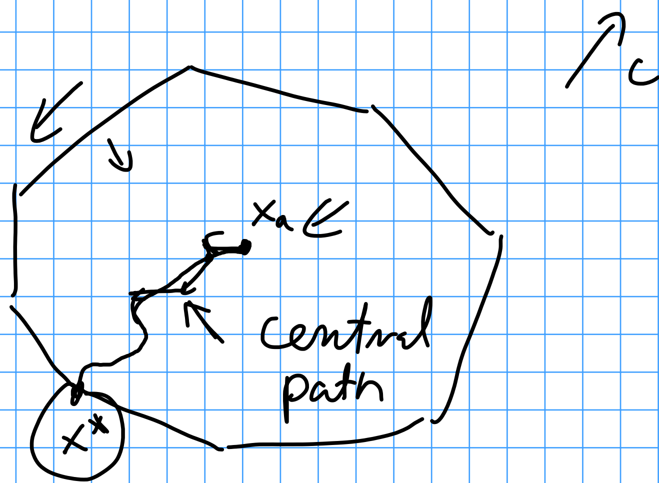
(2) $\lim_{\mu \to 0^+} x^*(\mu) = x^*$

Path following alg:

(1) Start with $x^*(\mu)$ for large μ

(2) Use $x^*(\mu) \rightarrow x^*(\mu')$ $\quad \mu' = (1-\delta)\mu$

Newton's method



x_a - analytic center of P

\# of iter. needed to get ε-approx

$$O\left(\sqrt{m}\,\underbrace{\log \frac{1}{\varepsilon}}\right)$$

$$\Rightarrow \tilde{O}\left(m^{\frac{3}{2}} \underbrace{\log \frac{1}{\varepsilon}}\right)$$

$$\Rightarrow \tilde{\Theta}\left(m^{\frac{10}{7}}\right)$$

$$O\left(m^{\frac{3}{7}} \log \frac{1}{\varepsilon}\right)$$

for unit-capacity max flo