Discussion Dec 11

Motivation for SVD is representing A which is $n \times d$ by a low-rank matrix A_k which is $n \times d$.

Objective is to minimize:

$$\sum_{i=1}^{n} A_i - a_i,$$

where $a_i = argmin_{v \in span(rows(A_k))} \|A_i - v\|_2$

Problem 2 (Part 1)

Show that for any matrix A we have $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$.

$$\|A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{d} a_{ij}^2 = \sum_{i=1}^{r} \sigma_i^2,$$

where r is rank of A.

Suffices to show that $\sigma_k^2 \leq \frac{\|A\|_F^2}{k}$. Suppose this is not the case then we have

$$\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2 > \|A\|_F^2$$

- $\|A\|_F^2 = \sum_{i=1}^{r} \sigma_i^2 = \|\sigma\|_2^2$
- $\|A\|_2 = \|\sigma\|_\infty$
- For square A we have $Tr(A) = \sum_{i=1}^{n} \lambda_i$

Problem 2 (Part 2)

Prove that for every A there exists a matrix B of rank at most k such that:

$$\|A - B\|_2 = \sigma_{k+1} \leq \frac{\|A\|_F}{\sqrt{k+1}}$$

Proof: pick $B = A_k$, where A_k is the best rank-k approximation for A constructed via SVD.

Problem 2 (Part 3)

Is it true that for every matrix A there exists a matrix B of rank at most k such that:

$$\|A - B\|_F \leq \frac{\|A\|_F}{\sqrt{k}}?$$

Let's take $A = I$. Then $\|A\|_F = \sqrt{n}$. We know that among all matrices B of rank at most k it holds that:

$$\|A - A_k\|_F \leq \|A - B\|_F,$$

where A_k is truncated SVD.

Problem 3

Let's construct a square matrix $A^T A$ of size $d \times d$. In lectures we suggest computing $(A^T A)^n$ for large enough n. If A is sparse this might not be taking a full advantage of sparsity.

Let's take a vector x and instead compute:

$$(A^T A)^n x$$

$$A^T A A^T (A \ldots (A^T (Ax)))$$

So we have $2n$ multiplications by a sparse matrix A which can be done in $nnz(A)$ time each, where $nnz(A)$ is the number of non-zero elements in A.